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LETTER TO THE EDITOR

On the upper critical dimension in Anderson localisation

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Abstract. We show that the anomalous scaling behaviour of the moments of the wavefunction at the threshold of Anderson localisation implies a log-normal distribution for the probability $|\psi|^2$ at first order in $\varepsilon = d-2$. We discuss this result and its implications critically and are led to conclude that the upper critical dimension of Anderson localisation is infinity.

In the present understanding of Anderson localisation a key role is played by the expansion of universal properties near the threshold in powers of the difference ε between the space dimensionality d and the lower critical dimension (LCD) of two. It was indeed argued by Abrahams *et al* (1979) that all quantum states of independent electrons in the presence of disorder should be marginally localised in two dimensions. Wegner (1979, 1980) showed how one could tackle the localisation problem by means of field theoretical techniques, in an analogous way to those introduced to discuss the critical behaviour of Heisenberg-like models near the LCD. In the latter cases, however, an ε expansion near the upper critical dimension (UCD), which equals four in ordinary critical phenomena, is also available. No such expansion is known at present for the localisation problem. Even the value d_c of the UCD has not yet been identified with certainty. Several arguments have been put forward (in particular by Kunz and Souillard (1983)) to support $d_c = 4$, although other values of d_c have been proposed.

We consider in this letter some arguments hinting towards the conclusion that d_c equals infinity, in the sense that mean-field behaviour is never observed. Our arguments hint at a first-order transition in the $d \to \infty$ limit, in agreement with the recent results of Efetov (1984) on the Cayley tree (it is not yet clear, however, whether Efetov's results directly imply a first-order phase transition).

The UCD can be identified (Toulouse 1974) as the value of d for which the hyperscaling relations (i.e. the d-dependent scaling laws) are satisfied by the classical exponents. Kunz and Souillard (1983) have argued that the classical exponents are those valid on the Cayley tree, for which they have obtained $\nu = \nu_0 = \frac{1}{2}$ for the exponent of the localisation length ξ and $\mu = \mu_0 = 1$ for the exponent of the conductivity σ. On the other hand, the scaling theories of localisation imply the scaling law

$$\mu = (d-2)\nu \tag{1}$$

which relates d, μ and ν. This expression is the analogue for localisation of the scaling law:

$$\gamma + 2\beta = d\nu \tag{2}$$

if one takes into account the fact that the exponent β equals zero for the localisation transition (since the order parameter is the density of states, which has no singular behaviour at the localisation threshold) and that $k^2\sigma$ (characterised by the exponent $\mu+2\nu$) is the analogue of the inverse susceptibility. Inserting the Cayley tree values ν_0, μ_0 (Last and Thouless 1974, Kunz and Souillard 1986) into (1), Kunz and Souillard (1983) obtain $d_c=4$. The same value of the UCD is obtained if, according to the same authors, we consider the values $\nu=1/\varepsilon$ and $\mu=1$ evaluated in the ε expansion as exact. Then $\nu=\nu_0$ and $\mu=\mu_0$ at $d=4$.

A third argument leading to $d_c=4$ has also been proposed by Kunz and Souillard (1983). They consider the behaviour near threshold of the inverse participation ratio $P^{(2)}(E)$, defined by

$$P^{(2)}(E)=\overline{\sum_\lambda |\psi_\lambda(\boldsymbol{r})|^4\delta(E-e_\lambda)}(\rho(E))^{-1} \tag{3}$$

where $\psi_\lambda(\boldsymbol{r})$ is an eigenfunction with eigenvalue e_λ of the Hamiltonian, $\rho(E)$ is the density of states and the average is over the realisations of the disorder. Wegner (1980) has shown that $P^{(2)}(E)$ near the localisation threshold E_c has the behaviour

$$P^{(2)}(E)\sim(E_c-E)^{\pi_2}\sim\xi^{-x_2} \tag{4}$$

where ξ is the localisation length and the exponent x_2 is given to first order in the ε expansion by

$$x_2=\pi_2/\nu=d-2\varepsilon. \tag{5}$$

If one accepts that there are no corrections to this expression in higher order in ε, one obtains $x_2=0$ at $d=4$, with a jump in the participation ratio at the mobility edge. This result, being in agreement with the corresponding result on the Cayley tree, again identifies $d_c=4$.

This last argument is subject to some criticisms. The inverse participation ratio is just the case $q=2$ of an infinite hierarchy of moments $P^{(q)}(E)$ of the probability density $|\psi|^2$. They are defined by

$$P^{(q)}(E)=\overline{\sum_\lambda |\psi_\lambda(\boldsymbol{r})|^{2q}\delta(E-e_\lambda)}(\rho(E))^{-1} \tag{6}$$

and they behave near E_c (according to Wegner 1980) as follows:

$$P^{(q)}(E)\sim\xi^{-x_q} \tag{7}$$

where the exponent x_q is given to first order in the ε expansion by

$$x_q=(q-1)d-q(q-1)\varepsilon. \tag{8}$$

There is no reason why the inverse participation ratio should be given a privileged role with respect to all these quantities. If we consider it in the case of the magnetic impurity problem, for instance, and we use the value of x_2 calculated by Pruisken (1985):

$$x_2=2-\sqrt{2\varepsilon} \tag{9}$$

we obtain $x_2=0$ at $d=4$, in contrast to the value $d_c=3$, obtained from the matching of the ε-expansion value of $\nu=1/2\varepsilon$ with the Cayley tree result $\nu=\frac{1}{2}$ (Kunz and Souillard 1983).

We can also argue that (5) cannot be extrapolated as such to $\varepsilon=2$. Given the meaning of $P^{(q)}$, the corresponding exponent x_q, considered as a function of q, should be non-decreasing. The approximate expression (8) can be accepted only until this is true, i.e. for $q-1<1/\varepsilon$, whereas it appears to vanish at the large value $q=d/\varepsilon$. It is unwise therefore to draw any conclusion from the location of the zeros of x_q.

It is more reasonable to consider what information the singular behaviour of the $P^{(q)}$ provides on the structure of the localisation threshold. If we assume that the sum appearing in (6) is dominated by a typical wavefunction φ_E (Castellani and Peliti 1986) we can interpret the properties of the $P^{(q)}$ in terms of properties of the probability density $|\varphi_E|^2$. Let us consider the probability distribution $w(z)$ of the values of $|\varphi_E|^2$:

$$w(z)=\sum_{r}\delta(z-|\varphi_E(\boldsymbol{r})|^2) \tag{10}$$

where the sum runs over the points of the lattice.

The hypothesis of the existence of a 'typical' wavefunction is in fact not necessary, if one accepts dealing instead with the quantity $\overline{w(z)}$ defined by

$$\overline{w(z)}=\overline{\sum_{\lambda}\delta(z-|\psi_\lambda(\boldsymbol{r})|^2)\delta(E-e_\lambda)}(\rho(E))^{-1}. \tag{10'}$$

We then obtain

$$P^{(q)}=\int \mathrm{d}z\, z^q w(z)\sim\xi^{-x_q}. \tag{11}$$

The expression (8) for the exponents x_q is compatible with a log-normal distribution of z:

$$z\,w(z)\sim\xi^d\exp\left(-\frac{(\ln z-\ln z_0)^2}{2\Delta^2}\right) \tag{12}$$

where

$$\Delta^2=2\varepsilon\ln\xi \tag{13}$$

$$\ln z_0=-(2+2\varepsilon)\ln\xi. \tag{14}$$

These results cannot be taken at face value, since we know that normalisability implies that $|\varphi_E|^2$ must be smaller than one at any lattice point and (12) has an unphysical tail towards higher values of z. When one computes the moment $P^{(q)}$ the integrand in (11) is a Gaussian in $\ln z$, peaked around the value $\ln z_0+\Delta^2 q$. For a sufficiently high value of q this peak will lie in the unphysical region. This corresponds to the region in which expression (8) becomes unacceptable. We can expect instead that x_q saturates at its maximum value $1/\varepsilon$ attained at $q=1+1/\varepsilon$. Then as $d\to\infty$ all the x_q $(q\geq 1)$ would tend simultaneously toward zero. We now try to interpret this behaviour in terms of the multifractal properties of the probability distribution $|\varphi_E|^2$.

Castellani and Peliti (1986) have attempted to relate the behaviour of the moments $P^{(q)}(E)$ to the multifractal character of $|\varphi_E(\boldsymbol{r})|^2$. In a multifractal the probability distribution $p(\boldsymbol{r})$ is given by a superposition of singularities of type α, each type of singularity being distributed over a fractal set of Hausdorff dimension $f(\alpha)$ (Benzi *et al* 1984, Halsey *et al* 1986). A singularity of type α is said to lie at the point $\boldsymbol{r}_0$ if, for $l\to 0$,

$$\int_{|\boldsymbol{r}-\boldsymbol{r}_0|<l}\mathrm{d}\boldsymbol{r}\,p(\boldsymbol{r})\sim l^\alpha. \tag{15}$$

It follows that if we consider a multifractal described with a resolution l, the moments $P^{(q)}$ of the probability distribution $p(r)$ can be given by

$$P^{(q)} \sim \int \mathrm{d}\alpha (l^{\alpha})^{q} l^{-f(\alpha)} \tag{16}$$

where the first factor takes care of the behaviour of p around the singularities, and the second factor takes care of the number of them present. From (16) by a saddle point integration we obtain, for $l \to 0$,

$$P^{(q)} \sim l^{\alpha^* q - f(\alpha^*)} \tag{17}$$

where α^* satisfies the equation

$$f'(\alpha^*) = q. \tag{18}$$

These concepts apply to the wavefunction if we identify $p(\boldsymbol{r})$ with $|\varphi_E(\boldsymbol{r})|^2$. We consider a probability coarse-grained over a box of linear size l:

$$p_l(r) = \int_{|\boldsymbol{r}'-\boldsymbol{r}|<l} \mathrm{d}\boldsymbol{r}' p(\boldsymbol{r}'). \tag{19}$$

Varying the resolution length l at a given value of the energy E corresponds to considering a varying localisation length $\xi' = \xi / l$ by scaling. The exponent in (17) may thus be identified with x_q. Therefore x_q and $f(\alpha)$ are related to each other by a Legendre transformation:

$$\alpha = \mathrm{d}x_q/\mathrm{d}q \quad f(\alpha) + x_q = \alpha q. \tag{20}$$

The Gaussian distribution of $\ln z$ has its counterpart in the Gaussian distribution of α: in fact, $f(\alpha)$ is given by the quadratic expression

$$f(\alpha) = d - \frac{1}{4\varepsilon}(d + \varepsilon - \alpha)^2. \tag{21}$$

Cutting the spurious tail at high values of $\ln z$ corresponds to assuming $\alpha \geq 0$. With this restriction, if we take the limit $d \to \infty$ in (21) we obtain

$$\lim_{d \to \infty} f(\alpha) = \alpha. \tag{22}$$

The integral in (16) is then dominated by the smallest value of α, i.e. zero. In this limit, therefore, all the $P^{(q)}$ become simultaneously constant as we had previously supposed. In this situation it is hard to expect the system to exhibit mean-field behaviour. One would tend more to attribute to it a first-order transition, compatible with the Cayley tree results of Efetov (1984).

The arguments we have reported hint towards a value of infinity for d_c, compatible with a situation in which the transition approaches a first-order transition as the dimensionality increases. It is indeed reasonable to expect that a system with a peculiar second-order transition for which $\beta = 0$ will more easily switch to a first-order one rather than to a mean-field behaviour with $\beta \neq 0$.

It would be interesting to evaluate higher-order terms in the expansion (8) of the indices x_q to find clues to its saturation at its maximum value. Numerical simulations could also help in checking the form we have suggested for the distribution of $|\varphi_E|^2$.

In conclusion we have found that the values of the probability density $|\varphi_E|^2$ are likely to be distributed according to a cutoff log-normal law. This implies that, as the dimensionality increases, the moments of the probability distribution all tend to become simultaneously constant. The $d \to \infty$ behaviour could then exhibit a first-order transition and mean-field behaviour would never be observed. The UCD of localisation would therefore be infinity.

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